

# Global regularity of 2D density patches for inhomogeneous Navier-Stokes

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## Abstract

This paper is about Lions' open problem on density patches [9]: whether inhomogeneous incompressible Navier-Stokes equations preserve the initial regularity of the free boundary given by density patches. Using the classical Sobolev spaces for the velocity, we establish the propagation of  $C^{1+\gamma}$  regularity with  $0 < \gamma < 1$  in the case of positive density.

**Keywords:** Navier-Stokes equations, density patch, global regularity.

## 1 Introduction

We consider the following incompressible inhomogeneous Navier-Stokes equations in the whole space  $\mathbb{R}^2$ :

$$\begin{cases} \partial_t \rho + u \cdot \nabla \rho = 0, \\ \rho(\partial_t u + u \cdot \nabla u) = \Delta u - \nabla p, \\ \nabla \cdot u = 0, \\ (\rho, u)|_{t=0} = (\rho_0, u_0), \end{cases} \quad (1)$$

where the unknowns  $\rho, u, p$  represent the density, velocity field and pressure of the fluid.

In the case where vacuum is allowed,  $\rho \geq 0$ , Simon [11] proved the global existence of weak solutions with finite energy. Afterwards, this result was extended to the case with variable viscosity by Lions in [9]. There, the author proposed the so-called *density patch problem*: assuming  $\rho_0 = 1_{D_0}$  for some domain  $D_0 \subset \mathbb{R}^2$ , the question is whether or not  $\rho(t) = 1_{D(t)}$  for some domain  $D(t)$  with the same regularity as the initial one. Theorem 2.1 in [9] ensures that the density remains as a patch preserving its volume, but gives no information about the persistence of regularity.

Previously to this problem, the analogous question in vortex patches in Euler equations arose great interest, due to the fact that several numerical results indicated the possible formation of finite time singularities. First Chemin [3] using paradifferential calculus and later Bertozzi and Constantin [2] by a geometrical harmonic analysis approach finally solved the *vortex patch problem* proving the contrary:  $C^{1+\gamma}$  vortex patches preserve their regularity in time.

Recently several results have appeared in the case of low regular positive density. First, Danchin and Mucha [4] showed the global well-posedness of (1) for initial densities allowing discontinuities across  $C^1$  interfaces with a sufficiently small jump and small initial velocity

in the Besov space  $B_{p,1}^{2/p-1}$ ,  $p \in [1, 4)$ . Later Paicu, Zhang and Zhang [10] obtained the global-wellposedness with initial data  $u_0 \in H^s$ ,  $s \in (0, 1)$  and initial positive density bounded from below and above removing the smallness conditions. Based on these results and using the techniques of striated regularity, Liao and Zhang [7], [8] have recently proved the persistence of  $W^{k,p}$  regularity,  $k \geq 3$ ,  $p \in (2, 4)$ , for initial patches of the form

$$\rho_0 = \rho_1 1_{D_0} + \rho_2 1_{D_0^c},$$

first assuming  $\rho_1, \rho_2 > 0$  close to each other, then for any pair of positive constants. By Sobolev embedding, this means that the boundary of the patch must be at least in  $C^{2+\gamma}$  for some  $\gamma > 0$ . Using the well-posedness result in [4], Danchin and Zhang [5] have recently obtained the propagation of  $C^{1+\gamma}$  patches for small jump and small  $u_0$  (also for large  $u_0$  but only locally in time).

In this paper we show that initial  $C^{1+\gamma}$  density patches preserve their regularity globally in time for any  $\rho_1, \rho_2 > 0$  and any  $u_0 \in H^{\gamma+s}$ ,  $s \in (0, 1 - \gamma)$ . As we noted in our previous work about Boussinesq temperature patches [6], the cancellation in the tangential direction to the patch is not needed to propagate low regularities. Starting with the estimates in [10], we will rewrite the equation as a forced heat equation to achieve the gain of two derivatives integrable in time. Although one cannot expect to get the needed regularity for the velocity in Sobolev spaces, we will take advantage of the fact that  $\rho$  remains as a patch with Lipschitz boundary. The extra difficulty due to the nonlinear appearance of the density, compared to the Boussinesq system, is overcome by noticing that the characteristic function of a Lipschitz patch belongs to the multiplier space  $\mathcal{M}(B_{\infty,\infty}^{1+\gamma})$  (see section below). Hence we will prove that  $u \in L^1(0, T; C^{1+\gamma})$  and thus the propagation follows by the particle trajectories system

$$\begin{cases} \frac{dX}{dt}(x, t) = u(X(x, t), t), \\ X(x, 0) = x. \end{cases}$$

Without considering regularity in the tangential direction to the density patch for the initial velocity, the initial conditions in [5], [8] are at the level of  $u_0 \in B_{2,1}^\gamma$ ,  $u_0 \in B_{2,1}^{1+\epsilon}$  ( $\epsilon > 0$ ), respectively. Indeed, as in [6], from the results of maximum regularity of the linear heat equation, we deem  $u_0 \in H^{\gamma+s}$  is sharp at the scale of Sobolev spaces from this approach.

## 2 Notation

In this section we include some definitions and notations used along the paper.

We will denote by  $(\partial_t - \Delta)_0^{-1} f$  the solution of the heat equation with force  $f$  and zero initial condition:

$$(\partial_t - \Delta)_0^{-1} f := \int_0^t e^{(t-\tau)\Delta} f(\tau) d\tau.$$

Above we use the standard notation  $e^{t\Delta} f = \mathcal{F}^{-1}(e^{-t|\xi|^2} \hat{f})$ , where  $\hat{\cdot}$  and  $\mathcal{F}^{-1}$  denote Fourier transform and its inverse.

We will say that  $\partial D(t) \in C^{1+\gamma}$  if there exists a parametrization of the boundary

$$\partial D(t) = \{z(\alpha, t) \in \mathbb{R}^2, \alpha \in [0, 1]\}$$

with  $z(t) \in C^{1+\alpha}$ .

We recall the definition of Besov spaces (see [1] for details). Consider the following Littlewood-Paley decomposition: let  $B = \{|\xi| \in \mathbb{R}^2 : |\xi| \leq 4/3\}$  and  $C = \{|\xi| \in \mathbb{R}^2 : 3/4 \leq |\xi| \leq 8/3\}$ , and fix two smooth radial functions  $\chi$  and  $\varphi$  supported in  $B$ ,  $C$ , respectively, and satisfying

$$\chi(\xi) + \sum_{j \geq 0} \varphi(2^{-j}\xi) = 1, \quad \forall \xi \in \mathbb{R}^2.$$

The nonhomogeneous dyadic blocks are defined as  $\Delta_j f = 0$  if  $j \leq -2$ ,  $\Delta_j f = \mathcal{F}^{-1}(\varphi(2^{-j}\xi)\hat{f}(\xi))$  for  $j \geq 0$  and  $\Delta_{-1}f = \mathcal{F}^{-1}(\chi(\xi)\hat{f}(\xi))$ . Then, the nonhomogeneous Besov spaces  $B_{p,q}^\gamma(\mathbb{R}^2)$ ,  $\gamma \in \mathbb{R}$ ,  $p, q \in [1, \infty]$  are defined by

$$B_{p,q}^\gamma(\mathbb{R}^2) = \{f \in S'(\mathbb{R}^2) : \|f\|_{B_{p,q}^\gamma} = \|2^{j\gamma} \|\Delta_j f\|_{L^p} \|_{l^q(\mathbb{Z})} < \infty\},$$

where  $S'(\mathbb{R}^2)$  denotes the space of tempered distributions over  $\mathbb{R}^2$ . We recall that  $H^s = B_{2,2}^s$  and  $C^{k+\gamma} = B_{\infty,\infty}^{k+\gamma}$  for  $s \in \mathbb{R}$ ,  $\gamma \in (0, 1)$ ,  $k \in \mathbb{N} \cup \{0\}$ .

Let  $E$  be a Banach space embedded in  $S'(\mathbb{R}^2)$ . We will use the spaces  $L^p(0, T; E)$  with norm  $\|f\|_{L_T^p(E)} := \|\|f\|_E\|_{L^p(0,T)}$ .

The *multiplier space*  $\mathcal{M}(E)$  is the set of functions  $\varphi$  such that  $\varphi f \in E$  for all  $f \in E$  and

$$\|\varphi\|_{\mathcal{M}(E)} := \sup_{\|f\|_E \leq 1} \|\varphi f\|_E < \infty.$$

### 3 Persistence of $C^{1+\gamma}$ regularity

We present below the theorem that establishes the propagation of regularity for  $C^{1+\gamma}$  patches in the case of positive density.

**Theorem 3.1.** *Assume  $\gamma \in (0, 1)$ ,  $s \in (0, 1 - \gamma)$ ,  $\rho_1, \rho_2 > 0$ . Let  $D_0 \subset \mathbb{R}^2$  be a bounded simply connected domain with boundary  $\partial D_0 \in C^{1+\gamma}$ ,  $u_0 \in H^{\gamma+s}$  a divergence-free vector field,*

$$\rho_0 = \rho_1 1_{D_0} + \rho_2 1_{D_0^c}, \quad (2)$$

*and  $1_{D_0}$  the characteristic function of  $D_0$ . Then, there exists a unique global solution  $(u, \rho)$  of (1) such that*

$$\rho(x, t) = \rho_1 1_{D(t)}(x) + \rho_2 1_{D(t)^c}(x) \text{ and } \partial D \in L^\infty(0, T; C^{1+\gamma}),$$

*where  $D(t) = X(D_0, t)$  with  $X$  the particle trajectories associated to the velocity field.*

*Moreover,*

$$u \in L^\infty(0, T; L^2) \cap L^p(0, T; H^{1+\gamma+\tilde{s}}) \cap L^1(0, T; C^{1+\gamma+\sigma}),$$

$$u_t \in L^q(0, T; H^{\gamma+\tilde{s}}),$$

*for any  $T > 0$ ,  $\sigma, \tilde{s} \in (0, s)$ ,  $p \in [1, 2/(1 - (s - \tilde{s}))]$  and  $q \in [1, 2/(2 - (s - \tilde{s}))]$ .*

**Proof:** First, as  $u_0 \in H^{\gamma+s}$  and  $0 < \min\{\rho_1, \rho_2\} < \rho_0 < \max\{\rho_1, \rho_2\} < \infty$ , the results in [10] yield the following estimates for any  $T \geq 0$ :

$$\begin{aligned} A_0(T) &\leq C(\|u_0\|_{L^2}), \\ A_1(T) &\leq C(\|u_0\|_{H^{\gamma+s}}), \\ A_2(T) &\leq C(\|u_0\|_{H^{\gamma+s}}), \end{aligned} \quad (3)$$

$$\int_0^T \|\nabla u\|_{L^\infty} dt \leq C(T, \|u_0\|_{H^{\gamma+s}}),$$

where the constant  $C$  also depends on  $\rho_1, \rho_2$ , and  $A_0, A_1, A_2$  are defined by

$$\begin{aligned} A_0(T) &= \sup_{[0,T]} \|\sqrt{\rho}u\|_{L^2}^2 + \int_0^T \|\nabla u\|_{L^2}^2 dt, \\ A_1(T) &= \sup_{[0,T]} \sigma(t)^{1-(\gamma+s)} \|\nabla u\|_{L^2}^2, \\ A_2(T) &= \sup_{[0,T]} \sigma(t)^{2-(\gamma+s)} (\|\sqrt{\rho}u_t\|_{L^2}^2 + \|\Delta u\|_{L^2}^2 + \|\nabla p\|_{L^2}^2) + \int_0^T \sigma(t)^{2-(\gamma+s)} \|\nabla u_t\|_{L^2}^2 dt, \end{aligned} \quad (4)$$

with  $\sigma(t) = \min\{1, t\}$ . Thus, by interpolation we get

$$\|u\|_{H^{1+\gamma+\tilde{s}}} \leq \|u\|_{H^1}^{1-\gamma-\tilde{s}} \|u\|_{H^2}^{\gamma+\tilde{s}} \leq c\sigma(t)^{-\frac{1-\gamma-s}{2}(1-\gamma-\tilde{s})-\frac{2-\gamma-s}{2}(\gamma+\tilde{s})},$$

and therefore

$$u \in L^p(0, T; H^{1+\gamma+\tilde{s}}), \quad p \in [1, 2/(1-(s-\tilde{s}))]. \quad (5)$$

Proceeding by interpolation again, it follows that

$$\int_0^T \|u_t\|_{H^{\gamma+\tilde{s}}}^q dt \leq \int_0^T \left( \|u_t\|_{L^2}^{1-\gamma-\tilde{s}} \|u_t\|_{H^1}^{\gamma+\tilde{s}} \right)^q dt \leq c \int_0^T \sigma(t)^{-\frac{2-\gamma-s}{2}(1-\gamma-\tilde{s})q} \frac{\|u_t\|_{H^1}^{q(\gamma+\tilde{s})} \sigma(t)^{\frac{2-\gamma-s}{2}(\gamma+\tilde{s})q}}{\sigma(t)^{\frac{2-\gamma-s}{2}(\gamma+\tilde{s})q}} dt,$$

hence by Hölder inequality we conclude that

$$u_t \in L^q(0, T; H^{\gamma+\tilde{s}}), \quad q \in [1, 2/(2-(s-\tilde{s}))]. \quad (6)$$

Notice that from (3) the velocity field satisfies  $u \in L^1(0, T; W^{1,\infty})$ , so the initial density is transported and remains as a patch

$$\rho(x, t) = \rho_1 1_{D(t)}(x) + \rho_2 1_{D(t)^c}$$

with Lipschitz boundary. It is known (see [12]) that the characteristic function of a Lipschitz bounded domain belongs to the multiplier space  $\mathcal{M}(B_{p,q}^s)$  if and only if  $-1 + \frac{1}{p} < s < \frac{1}{p}$ , where  $p, q \in [1, \infty]$ . Therefore we deduce that

$$\rho \in L^\infty(0, T; \mathcal{M}(B_{\infty,\infty}^{-1+\gamma+s})). \quad (7)$$

We will use this fact combined with the smoothing properties of the heat equation to prove that the velocity field is indeed more regular, specifically, we will show that  $u \in L^1(0, T; C^{1+\gamma+\sigma})$ . Rewrite the momentum equation in (1) as a forced heat equation

$$u_t - \Delta u = -\rho \nabla \cdot (u \otimes u) + (1 - \rho)u_t - \nabla p.$$

We apply first the Leray projector  $\mathbb{P} = \text{Id} - \nabla \Delta^{-1}(\nabla \cdot)$  to obtain

$$u_t - \Delta u = -\mathbb{P}(\rho \nabla \cdot (u \otimes u)) + \mathbb{P}((1 - \rho)u_t), \quad (8)$$

and denote

$$\begin{aligned} u &= v_1 + v_2 + v_3, \\ v_1 &= e^{t\Delta} u_0, \quad v_2 = -(\partial_t - \Delta)_0^{-1} \mathbb{P}(\rho \nabla \cdot (u \otimes u)), \quad v_3 = (\partial_t - \Delta)_0^{-1} \mathbb{P}((1 - \rho)u_t). \end{aligned} \quad (9)$$

Since  $u_0 \in H^{\gamma+s}$ , classic estimates of the homogeneous heat equation yields

$$v_1 \in L^1(0, T; H^{2+\gamma+\tilde{s}}) \hookrightarrow L^1(0, T; C^{1+\gamma+\tilde{s}}). \quad (10)$$

Next, we will use the following heat equation estimates in negative Hölder spaces.

**Lemma 3.2.** *Let  $f \in L^p(0, T; B_{\infty, \infty}^{-1+\gamma+\tilde{s}})$ ,  $p \in [1, \infty]$ ,  $\gamma + \tilde{s} \in (0, 1)$ ,  $\sigma \in (0, \tilde{s})$ . If in addition  $f \in L^p(0, T; L^2)$  or  $f \in L^p(0, T; L^1)$ , then it holds that*

$$\|(\partial_t - \Delta)_0^{-1} f\|_{L_T^p(B_{\infty, \infty}^{1+\gamma+\sigma})} \leq c\|f\|_{L_T^p(B_{\infty, \infty}^{-1+\gamma+\tilde{s}})} + c\|f\|_{L_T^p(L^2)}, \quad (11)$$

$$\|(\partial_t - \Delta)_0^{-1} f\|_{L_T^p(B_{\infty, \infty}^{1+\gamma+\sigma})} \leq c\|f\|_{L_T^p(B_{\infty, \infty}^{-1+\gamma+\tilde{s}})} + c\|f\|_{L_T^p(L^1)}, \quad (12)$$

Proof: We use the definition of Besov spaces to split into low and high frequencies:

$$\begin{aligned} \|(\partial_t - \Delta)_0^{-1} f\|_{B_{\infty, \infty}^{1+\gamma+\sigma}} &= 2^{-(1+\gamma+\sigma)} \|\mathcal{F}^{-1}(\chi(\xi) \mathcal{F}((\partial_t - \Delta)_0^{-1} f))\|_{L^\infty} \\ &\quad + \sup_{j \geq 0} 2^{j(1+\gamma+\sigma)} \|\Delta_j((\partial_t - \Delta)_0^{-1} f)\|_{L^\infty} = F_1 + F_2. \end{aligned}$$

By the decay properties of the heat kernel, we obtain

$$F_2 \leq \sup_{j \geq 0} \int_0^t 2^{j(2-(\tilde{s}-\sigma))} e^{-c(t-\tau)2^{2j}} 2^{j(-1+\gamma+\tilde{s})} \|\Delta_j f\|_{L^\infty}(\tau) d\tau \leq \int_0^t \frac{c}{(t-\tau)^{1-(\tilde{s}-\sigma)/2}} \|f\|_{B_{\infty, \infty}^{-1+\gamma+\tilde{s}}} d\tau,$$

hence by Young's convolution inequality it follows that

$$\|F_2\|_{L_T^p} \leq c\|f\|_{L_T^p(B_{\infty, \infty}^{-1+\gamma+\tilde{s}})}.$$

We continue to deal with the low frequencies:

$$F_1 \leq c \int_0^t \|\chi(\xi) e^{-(t-\tau)|\xi|^2} \hat{f}(\xi, \tau)\|_{L^1} d\tau.$$

Now, if  $f \in L^p(0, T; L^1)$ , we proceed as follows:

$$\|F_1\|_{L_T^p} \leq \left\| \int_0^t \|\hat{f}(\xi, \tau)\|_{L^\infty} \|\chi(\xi) e^{-(t-\tau)|\xi|^2}\|_{L^1} d\tau \right\|_{L_T^p} \leq c(T) \|f\|_{L_T^p(L^1)},$$

while if  $f \in L^p(0, T; L^2)$ , as  $\chi(\xi)$  is supported in a ball, we obtain that

$$\|F_1\|_{L_T^p} \leq \left\| \int_0^t \|\chi(\xi) e^{-(t-\tau)|\xi|^2} \hat{f}(\xi, \tau)\|_{L^2} d\tau \right\|_{L_T^p} \leq \left\| \int_0^t \|\hat{f}(\xi, \tau)\|_{L^2} d\tau \right\|_{L_T^p} \leq c(T) \|f\|_{L_T^p(L^2)}.$$

□

From (5) we deduce  $\nabla \cdot (u \otimes u) \in L^{p/2}(0, T; H^{\gamma+\tilde{s}}) \hookrightarrow L^{p/2}(0, T; B_{\infty, \infty}^{-1+\gamma+\tilde{s}})$ . In addition, notice that

$$\|\rho u \cdot \nabla u\|_{L_T^{p/2}(L^1)} \leq \|\rho\|_{L_T^\infty(L^\infty)} \|u\|_{L_T^\infty(L^2)} \|\nabla u\|_{L_T^{p/2}(L^2)}.$$

Taking into account (7) and the fact that we can choose  $p$  such that  $p/2 \geq 1$ , it follows from (12) that

$$(\partial_t - \Delta)_0^{-1}(\rho u \cdot \nabla u) \in L^1(0, T; C^{1+\gamma+\sigma}).$$

Recalling that the Leray projector is a homogeneous Fourier multiplier of degree zero, we conclude that  $v_2$  in (9) satisfies

$$v_2 \in L^1(0, T; C^{1+\gamma+\sigma}). \quad (13)$$

Using (6) and (7) we get  $\rho u_t \in L^q(0, T; B_{\infty, \infty}^{-1+\gamma+\tilde{s}})$  for some  $q \geq 1$ . As  $\rho u_t \in L^\infty(0, T; L^2)$ , we obtain from (11) as before that

$$v_3 \in L^1(0, T; C^{1+\gamma+\sigma}). \quad (14)$$

Finally, from (9), (10), (13) and (14), we conclude that  $u \in L^1(0, T; C^{1+\gamma+\sigma})$ . Hence, using the particle trajectories system (1) and Grönwall's inequality we obtain

$$\|\nabla X\|_{C^\gamma} \leq \|\nabla X_0\|_{C^\gamma} e^{\int_0^t \|\nabla u\|_{L^\infty} d\tau} + \int_0^t \|\nabla u(\tau)\|_{C^\gamma} \|\nabla X(\tau)\|_{L^\infty}^{1+\gamma} e^{\int_\tau^t \|\nabla u\|_{L^\infty} ds} d\tau,$$

which yields the persistence of  $C^{1+\gamma}$  regularity of the density patch  $\|z\|_{L^\infty(0, T; C^{1+\gamma})} \leq C(T)$ . As  $0 < \tilde{s} < s$  is arbitrary, following the same lines it is easy to prove the case  $\tilde{s} \leq \sigma$ .  $\square$

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